



# THE FORCED VIBRATION AND BOUNDARY CONTROL OF PRETWISTED TIMOSHENKO BEAMS WITH GENERAL TIME DEPENDENT ELASTIC BOUNDARY CONDITIONS

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*(Received 21 September 2000, and in final form 18 October 2001)*

The governing differential equations and the general time-dependent elastic boundary conditions for the coupled bending–bending forced vibration of a pretwisted non-uniform Timoshenko beam are derived by Hamilton’s principle. By introducing a general change of dependent variable with shifting functions, the original system is transformed into a system composed of four non-homogeneous governing differential equations and eight homogeneous boundary conditions. The transformed system is proved to be self-adjoint. Consequently, the method of separation of variables can be used to solve the transformed problem. The physical meanings of these shifting functions are explored. The orthogonality condition for the eigenfunctions of a pretwisted non-uniform beam with elastic boundary conditions is also derived. The relation between the shifting functions and the stiffness matrix is derived. The boundary control of a pretwist Timoshenko beam is studied. The effects of the total pretwist angle, the position of loading and the boundary spring constants on the energy required to control the performance of a pretwisted beam are investigated.

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## 1. INTRODUCTION

The dynamic analysis of the pretwisted beams is important in the design of a number of engineering components, e.g., turbine blades, helicopter rotor blades and gear teeth. An interesting review of the subject can be found in the review paper by Rosen [1]. The forced vibration problem of a pretwisted non-uniform beam with general elastic time-dependent boundary conditions is common in engineering applications. Thus, it is necessary to develop an accurate and simple method to solve this complicated problem and to find its performance.

The vibrations of unpretwisted uniform Bernoulli–Euler beams with classical time-dependent boundary conditions can be solved by using the method of Laplace transform [2] and the method of Mindlin–Goodman [3, 4]. In the Mindlin–Goodman method, a change of dependent variable together with four shifting polynomial functions of the fifth order is introduced. In general, by properly selecting these shifting polynomial functions,

the original system will be transformed into a system composed of a non-homogeneous governing differential equation with four homogeneous boundary conditions. Consequently, the method of separation of variables can be used to solve the problem. Lee and Lin [5] gave the dynamic analysis of a non-uniform Bernoulli–Euler beam with general time-dependent boundary conditions. They generalized the method of Mindlin–Goodman and introduced four shifting functions with the physical meaning instead of those functions with no physical meaning given by Mindlin and Goodman [3]. The vibrations of unpretwisted uniform Timoshenko beams with classical time-dependent boundary conditions were studied by Herrmann [6] and Berry and Nagdhi [7] by using the method of Mindlin–Goodman. Lee and Lin [8] extended the previous study carried out by Lee and Lin [5] and further generalized the method of Mindlin–Goodman to develop a solution procedure for studying the vibrations of an unpretwisted non-uniform Timoshenko beams with general time-dependent boundary conditions.

Approximation methods are very useful tools to investigate the free vibrations of pretwisted beams where exact solutions are difficult to obtain even for the simplest cases. For Bernoulli–Euler beams, Dawson [9], Dawson and Carnegie [10] used the Rayleigh–Ritz method and transformation techniques to study the effects of uniform pretwist on the frequencies of cantilever blades. Carnegie and Thomas [11] and Rao [12, 13] used the Rayleigh–Ritz method and Ritz–Galerkin method to study the effects of uniform pretwist and the taper ratio on the frequencies of cantilever blades respectively. Sabuncu [14] found by using the finite element method that the effect of trigonometric pretwist angle on the frequencies increased as the pretwist angle increased. Rosard and Lestar [15] and Rao and Carnegie [16] used the transfer matrix method to determine the frequencies of vibration of the cantilever beam with uniform pretwist. Rosard and Lestar [15] assumed that the displacements at each element are linear. Rao and Carnegie [16] used an iteration procedure to determine the displacements at each element while the initial displacements were assumed to be linear. The difficulties of the methods given by Rosard and Lestar [15] and Rao and Carnegie [16] are overcome by Lin [17]. Lin [17] presented a simple and accurate transfer matrix method for an elastically restrained non-uniform beam with arbitrary pretwist. Moreover, it was found that the influence of the pretwist angle on the natural frequencies of the beam with non-uniform pretwist is greater than on those of the beam with uniform pretwist. The influence of the pretwist angle on the natural frequencies of higher modes are greater than on those of lower modes. The stiffer the boundary supports, the greater is the influence of the pretwist angle on the natural frequencies. Lin [18] studied the force vibration of an elastically restrained non-uniform beam with time-dependent boundary conditions.

For pretwisted Timoshenko beams, the influence of the shear deformation and the rotatory inertia have been considered. Carnegie [19] determined the fundamental frequency of a cantilever beam by using Rayleigh’s principle. Dawson *et al.* [20] used the transformation method to study the effects of shear deformation and rotatory inertia on the natural frequencies. Gupta and Rao [21] and Abbas [22] used the finite element method to determine the natural frequencies of uniformly pretwisted tapered cantilever blading. Subrahmanyam *et al.* [23] and Subrahmanyam and Rao [24] used the finite element method and the Reissner method to determine the natural frequencies of uniformly pretwisted tapered cantilever blading respectively. Celep and Turhan [25] used the Galerkin method to investigate the influence of non-uniform pretwisting on the natural frequencies of uniform cross-sectional cantilever or simply supported beams. Lin *et al.* [26] derived the exact field transfer matrix of a non-uniform pretwisted Timoshenko beam with arbitrary pretwist and studied the free vibration of a pretwisted Timoshenko beam with the elastic boundary conditions. No research has been devoted to the forced vibration and the

boundary control of the pretwisted Timoshenko beam with time-dependent elastic boundary conditions.

In this paper, the governing differential equations and the general time-dependent elastic boundary conditions for the coupled bending–bending forced vibration of a pretwisted Timoshenko beam are derived by Hamilton’s principle. A solution procedure for studying the dynamic behavior of the system is developed by using the method of Mindlin–Goodman and the eigensolutions of the system obtained by using the modified transfer method given by Lin *et al.* [26]. A general change of dependent variable with shifting functions is introduced and the physical meanings of these shifting functions are further explored. The orthogonality condition for the eigenfunctions of a non-uniform pretwisted beam with elastic boundary conditions is also derived. The stiffness matrix for a non-uniform beam with arbitrary pretwist is derived. The relation between the shifting functions and the stiffness matrix is derived. The vibration control of a pretwisted beam with boundary inputs is investigated.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

Consider the forced vibration problem of a pretwisted non-uniform Timoshenko beam with time-dependent elastic boundary conditions as shown in Figure 1. Both shear deformation and rotatory inertia are considered. The displacement fields of the beam are

$$\begin{aligned}
 u(x, t) &= -z\Psi_z(x, t) - y\Psi_y(x, t), \quad v(x, t) = v(x, t), \\
 w(x, t) &= w(x, t)
 \end{aligned}
 \tag{1}$$

where  $u$ ,  $v$ , and  $w$  are the displacements in the  $x$ ,  $y$ , and  $z$  directions respectively.  $\Psi_y$  and  $\Psi_z$  are the angle of rotation due to bending about the  $z$  and  $y$  directions respectively.  $t$  is the time variable. The total potential energy  $\Pi$  and the kinetic energy  $\bar{K}$  of beam are

$$\begin{aligned}
 \Pi &= \frac{1}{2} \int_0^L \int_A (\sigma_{xx}\epsilon_{xx} + 2\sigma_{xy}\epsilon_{xy} + 2\sigma_{xz}\epsilon_{xz}) dA dx + \frac{1}{2} K_{z\theta L} [\Psi_z(0, t) - f_1(t)]^2 \\
 &+ \frac{1}{2} K_{zTL} [w(0, t) - f_2(t)]^2 + \frac{1}{2} K_{y\theta L} [\Psi_y(0, t) - f_3(t)]^2
 \end{aligned}$$

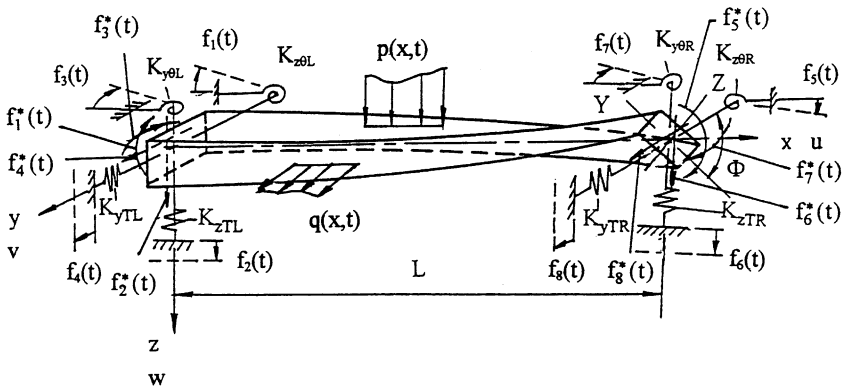


Figure 1. Geometry and coordinate system of a pretwisted beam subjected to the transverse loads and the boundary excitations.

$$\begin{aligned}
& + \frac{1}{2} K_{yTL} [v(0, t) - f_4(t)]^2 + \frac{1}{2} K_{z\theta R} [\Psi_z(L, t) - f_5(t)]^2 \\
& + \frac{1}{2} K_{zTR} [w(L, t) - f_6(t)]^2 + \frac{1}{2} K_{y\theta R} [\Psi_y(L, t) - f_7(t)]^2 \\
& + \frac{1}{2} K_{yTR} [v(L, t) - f_8(t)]^2 - \int_0^L [p(x, t)w(x, t) + q(x, t)v(x, t)] dx \\
& - f_1^*(t)\Psi_z(0, t) - f_2^*(t)w(0, t) - f_3^*(t)\Psi_y(0, t) - f_4^*(t)v(0, t) \\
& - f_5^*(t)\Psi_z(L, t) - f_6^*(t)w(L, t) - f_7^*(t)\Psi_y(L, t) - f_8^*(t)v(L, t), \tag{2}
\end{aligned}$$

$$\bar{K} = \frac{1}{2} \int_0^L \int_A [(\partial w / \partial t)^2 + (\partial v / \partial t)^2 + (\partial u / \partial t)^2] \rho \, dA \, dx, \tag{3}$$

where  $A$  is the cross-sectional area of beam.  $E$  is Young's modulus.  $f_i$  and  $f_i^*$ ,  $i = 1, 2, \dots, 8$ , are the slopes, displacements, external moments, and shear excitations at the left and right of the beam in the  $y$  and  $z$  directions respectively.  $p(x, t)$  and  $q(x, t)$  are the external transverse loads in the  $z$  and  $y$  directions respectively.  $L$  is the length of beam.  $\rho$  is the mass density per unit volume.  $\sigma$  and  $\varepsilon$  are the stress and the strain respectively.  $K$  denotes a spring constant. The subscripts  $y$  and  $z$  denote the  $y$  and  $z$  directions respectively. The subscripts  $T$  and  $\theta$  denote the translational and rotational springs respectively. The subscripts  $L$  and  $R$  denote the left end and the right end of the beam. Application of Hamilton's principle yields the coupled governing differential equations and the associated time-dependent elastic boundary conditions.

In terms of the following dimensionless quantities

$$B_{ij}(\xi) = \frac{E(x)I_{ij}(x)}{E(0)I_{yy}(0)}, \quad F_i = f_i, \quad i = 1, 3, 5, 7, \quad F_i = \frac{f_i}{L}, \quad i = 2, 4, 6, 8,$$

$$F_i^* = \frac{f_i^*L}{E(0)I_{yy}(0)}, \quad i = 1, 3, 5, 7, \quad F_i^* = \frac{f_i^*L^2}{E(0)I_{yy}(0)}, \quad i = 2, 4, 6, 8,$$

$$\bar{F}_i(\tau) = \gamma_{i1}F_i(\tau) + \gamma_{i2}F_i^*(\tau), \quad m(\xi) = \frac{\rho(x)A(x)}{\rho(0)A(0)}, \quad P(\xi, \tau) = \frac{p(x, t)L^3}{E(0)I_{yy}(0)},$$

$$Q(\xi, \tau) = \frac{q(x, t)L^3}{E(0)I_{yy}(0)}, \quad R_{ij}(\xi) = \frac{\rho(x)I_{ij}(x)}{\rho(0)I_{yy}(0)}, \quad S(\xi) = \frac{\kappa(x)G(x)A(x)}{\kappa(0)G(0)A(0)},$$

$$V(\xi) = \frac{v(x)}{L}, \quad W(\xi) = \frac{w(x)}{L}, \quad \beta_1 = \frac{K_{z\theta L}L^3}{E(0)I_{yy}(0)},$$

$$\beta_2 = \frac{K_{zTL}L^3}{E(0)I_{yy}(0)}, \quad \beta_3 = \frac{K_{y\theta L}L}{E(0)I_{yy}(0)}, \quad \beta_4 = \frac{K_{yTL}L^3}{E(0)I_{yy}(0)},$$

$$\begin{aligned}
\beta_5 &= \frac{K_{z0}L}{E(0)I_{yy}(0)}, \quad \beta_5 = \frac{K_{zTL}L^3}{E(0)I_{yy}(0)}, \quad \beta_7 = \frac{K_{y0}L}{E(0)I_{yy}(0)}, \\
\beta_8 &= \frac{K_{yTR}L^3}{E(0)I_{yy}(0)}, \quad \gamma_{i1} = \frac{\beta_i}{1 + \beta_i}, \quad \gamma_{i2} = \frac{1}{1 + \beta_i}, \\
\eta &= \frac{I_{yy}(0)}{A(0)L^2}, \quad \mu = \frac{E(0)I_{yy}(0)}{\kappa(0)G(0)A(0)L^2}, \quad \xi = \frac{x}{L}, \\
\tau &= \frac{t}{L^2} \sqrt{\frac{E(0)I_{yy}(0)}{\rho(0)A(0)}}, \tag{4}
\end{aligned}$$

the four coupled dimensionless governing characteristic differential equations of motion are obtained:

$$-\frac{\partial}{\partial \xi} \left[ \frac{S(\xi)}{\mu} \left( \frac{\partial W}{\partial \xi} - \Psi_z \right) \right] + m(\xi) \frac{\partial^2 W}{\partial \tau^2} = P(\xi, \tau), \tag{5}$$

$$\begin{aligned}
\frac{\partial}{\partial \xi} \left[ B_{yy} \frac{\partial \Psi_z}{\partial \xi} \right] + \frac{\partial}{\partial \xi} \left[ B_{yz} \frac{\partial \Psi_y}{\partial \xi} \right] + \frac{S(\xi)}{\mu} \left( \frac{\partial W}{\partial \xi} - \Psi_z \right) \\
- \eta R_{yy}(\xi) \frac{\partial^2 \Psi_z}{\partial \tau^2} - \eta R_{yz}(\xi) \frac{\partial^2 \Psi_y}{\partial \tau^2} = 0, \tag{6}
\end{aligned}$$

$$-\frac{\partial}{\partial \xi} \left[ \frac{S(\xi)}{\mu} \left( \frac{\partial V}{\partial \xi} - \Psi_y \right) \right] + m(\xi) \frac{\partial^2 V}{\partial \tau^2} = Q(\xi, \tau), \tag{7}$$

$$\begin{aligned}
\frac{\partial}{\partial \xi} \left[ B_{yz} \frac{\partial \Psi_z}{\partial \xi} \right] + \frac{\partial}{\partial \xi} \left[ B_{zz} \frac{\partial \Psi_y}{\partial \xi} \right] + \frac{S(\xi)}{\mu} \left( \frac{\partial V}{\partial \xi} - \Psi_y \right) \\
- \eta R_{yz}(\xi) \frac{\partial^2 \Psi_z}{\partial \tau^2} - \eta R_{zz}(\xi) \frac{\partial^2 \Psi_y}{\partial \tau^2} = 0, \quad \xi \in (0, 1) \tag{8}
\end{aligned}$$

and the associated dimensionless elastic boundary conditions are at  $\xi = 0$ :

$$-\gamma_{12} \left( B_{yy} \frac{\partial \Psi_z}{\partial \xi} + B_{yz} \frac{\partial \Psi_y}{\partial \xi} \right) + \gamma_{11} \Psi_z = \bar{F}_1(\tau), \tag{9}$$

$$-\gamma_{22} \frac{S}{\mu} \left( \frac{\partial W}{\partial \xi} - \Psi_z \right) + \gamma_{21} W = \bar{F}_2(\tau), \tag{10}$$

$$-\gamma_{32} \left( B_{yz} \frac{\partial \Psi_z}{\partial \xi} + B_{zz} \frac{\partial \Psi_y}{\partial \xi} \right) + \gamma_{31} \Psi_y = \bar{F}_3(\tau), \tag{11}$$

$$-\gamma_{42} \frac{S}{\mu} \left( \frac{\partial V}{\partial \xi} - \Psi_y \right) + \gamma_{41} V = \bar{F}_4(\tau), \tag{12}$$

At  $\xi = 1$ :

$$\gamma_{52} \left( B_{yy} \frac{\partial \Psi_z}{\partial \xi} + B_{yy} \frac{\partial \Psi_y}{\partial \xi} \right) + \gamma_{51} \Psi_z = \bar{F}_5(\tau), \quad (13)$$

$$\gamma_{62} \frac{S}{\mu} \left( \frac{\partial W}{\partial \xi} - \Psi_z \right) + \gamma_{61} W = \bar{F}_6(\tau), \quad (14)$$

$$\gamma_{72} \left( B_{yz} \frac{\partial \Psi_z}{\partial \xi} + B_{zz} \frac{\partial \Psi_y}{\partial \xi} \right) + \gamma_{71} \Psi_z = \bar{F}_7(\tau), \quad (15)$$

$$\gamma_{82} \frac{S}{\mu} \left( \frac{\partial V}{\partial \xi} - \Psi_y \right) + \gamma_{81} V = \bar{F}_8(\tau), \quad (16)$$

where  $I_{ij}$ ,  $i = j$  and  $i \neq j$ , are the area moment of inertia and the product of inertia respectively.

When the dimensionless translational spring constant is infinity or zero, the time-dependent displacement or the time-dependent shear force is prescribed. If the dimensionless rotational spring constant is infinity or zero, the time-dependent slope or the time-dependent moment is prescribed.

The associated dimensionless initial conditions of the motion are

$$\begin{aligned} W(\xi, 0) &= W_0(\xi), & V(\xi, 0) &= V_0(\xi), \\ \frac{\partial W(\xi, 0)}{\partial \tau} &= \dot{W}_0(\xi), & \frac{\partial V(\xi, 0)}{\partial \tau} &= \dot{V}_0(\xi), \\ \Psi_z(\xi, 0) &= \Psi_{z0}(\xi), & \Psi_y(\xi, 0) &= \Psi_{y0}(\xi), \\ \frac{\partial \Psi_z(\xi, 0)}{\partial \tau} &= \dot{\Psi}_{z0}(\xi), & \frac{\partial \Psi_y(\xi, 0)}{\partial \tau} &= \dot{\Psi}_{y0}(\xi). \end{aligned} \quad (17)$$

### 3. SOLUTION METHOD

#### 3.1. CHANGE OF VARIABLE

To find the solution for these differential equations with variable coefficients and non-homogeneous elastic boundary conditions, one generalizes the method given by Lin [18] by taking

$$\begin{aligned} W(\xi, \tau) &= \bar{w}(\xi, \tau) + \sum_{i=1}^8 \bar{F}_i(\tau) g_i(\xi), & V(\xi, \tau) &= \bar{v}(\xi, \tau) + \sum_{i=1}^8 \bar{F}_i(\tau) \bar{g}_i(\xi), \\ \Psi_z(\xi, \tau) &= \bar{\varphi}_z(\xi, \tau) + \sum_{i=1}^8 \bar{F}_i(\tau) h_i(\xi), & \Psi_y(\xi, \tau) &= \bar{\varphi}_y(\xi, \tau) + \sum_{i=1}^8 \bar{F}_i(\tau) \bar{h}_i(\xi), \end{aligned} \quad (18)$$

where the shifting functions  $g_i(\xi)$ ,  $\bar{g}_i(\xi)$ ,  $h_i(\xi)$  and  $\bar{h}_i(\xi)$  are chosen to satisfy the following differential equations:

$$\frac{d}{d\xi} \left[ \frac{S(\xi)}{\mu} \left( \frac{dg_i}{d\xi} - h_i \right) \right] = 0, \quad (19)$$

$$\frac{d}{d\xi} \left[ B_{yy} \frac{dh_i}{d\xi} \right] + \frac{d}{d\xi} \left[ B_{yz} \frac{d\bar{h}_i}{d\xi} \right] + \frac{S(\xi)}{\mu} \left( \frac{dg_i}{d\xi} - h_i \right) = 0, \quad (20)$$

$$\frac{d}{d\xi} \left[ \frac{S(\xi)}{\mu} \left( \frac{d\bar{g}_i}{d\xi} - \bar{h}_i \right) \right] = 0, \quad (21)$$

$$\frac{d}{d\xi} \left[ B_{yz} \frac{dh_i}{d\xi} \right] + \frac{d}{d\xi} \left[ B_{zz} \frac{d\bar{h}_i}{d\xi} \right] + \frac{S(\xi)}{\mu} \left( \frac{d\bar{g}_i}{d\xi} - \bar{h}_i \right) = 0, \quad i = 1, 2, \dots, 8 \quad (22)$$

and the associated dimensionless elastic boundary conditions.

At  $\xi = 0$ :

$$-\gamma_{12} \left( B_{yy} \frac{dh_i}{d\xi} + B_{yz} \frac{d\bar{h}_i}{d\xi} \right) + \gamma_{11} h_i = \delta_{i1}, \quad (23)$$

$$-\gamma_{22} \frac{S}{\mu} \left( \frac{dg_i}{d\xi} - h_i \right) + \gamma_{21} g_i = \delta_{i2}, \quad (24)$$

$$-\gamma_{32} \left( B_{yz} \frac{dh_i}{d\xi} + B_{zz} \frac{d\bar{h}_i}{d\xi} \right) + \gamma_{31} \bar{h}_i = \delta_{i3}, \quad (25)$$

$$-\gamma_{42} \frac{S}{\mu} \left( \frac{d\bar{g}_i}{d\xi} - \bar{h}_i \right) + \gamma_{41} \bar{g}_i = \delta_{i4}, \quad (26)$$

At  $\xi = 1$ :

$$\gamma_{52} \left( B_{yy} \frac{dh_i}{d\xi} + B_{yz} \frac{d\bar{h}_i}{d\xi} \right) + \gamma_{51} h_i = \delta_{i5}, \quad (27)$$

$$\gamma_{62} \frac{S}{\mu} \left( \frac{dg_i}{d\xi} - h_i \right) + \gamma_{61} g_i = \delta_{i6}, \quad (28)$$

$$\gamma_{72} \left( B_{yz} \frac{dh_i}{d\xi} + B_{zz} \frac{d\bar{h}_i}{d\xi} \right) + \gamma_{71} \bar{h}_i = \delta_{i7}, \quad (29)$$

$$\gamma_{82} \frac{S}{\mu} \left( \frac{d\bar{g}_i}{d\xi} - \bar{h}_i \right) + \gamma_{81} \bar{g}_i = \delta_{i8}, \quad (30)$$

where  $\delta_{ij}$  is a Kronecker symbol.

After substituting equations (18–30) into equations (5–16), one obtains the following differential equations in terms of  $\bar{w}(\xi, \tau)$ ,  $\bar{v}(\xi, \tau)$ ,  $\bar{\varphi}_z(\xi, \tau)$  and  $\bar{\varphi}_y$ :

$$-\frac{\partial}{\partial \xi} \left[ \frac{S(\xi)}{\mu} \left( \frac{\partial \bar{w}}{\partial \xi} - \bar{\varphi}_z \right) \right] + m(\xi) \frac{\partial^2 \bar{w}}{\partial \tau^2} = \bar{p}(\xi, \tau), \quad (31)$$

$$\begin{aligned} & \frac{\partial}{\partial \xi} \left[ B_{yy} \frac{\partial \bar{\varphi}_z}{\partial \xi} \right] + \frac{\partial}{\partial \xi} \left[ B_{yz} \frac{\partial \bar{\varphi}_y}{\partial \xi} \right] + \frac{S(\xi)}{\mu} \left( \frac{\partial \bar{w}}{\partial \xi} - \bar{\varphi}_z \right) \\ & - \eta R_{yy}(\xi) \frac{\partial^2 \bar{\varphi}_z}{\partial \tau^2} - \eta R_{yz}(\xi) \frac{\partial^2 \bar{\varphi}_y}{\partial \tau^2} = \tilde{p}(\xi, \tau), \end{aligned} \quad (32)$$

$$-\frac{\partial}{\partial \xi} \left[ \frac{S(\xi)}{\mu} \left( \frac{\partial \bar{v}}{\partial \xi} - \bar{\varphi}_y \right) \right] + m(\xi) \frac{\partial^2 \bar{v}}{\partial \tau^2} = \bar{q}(\xi, \tau), \quad (33)$$

$$\begin{aligned} & \frac{\partial}{\partial \xi} \left[ B_{yz} \frac{\partial \bar{\varphi}_z}{\partial \xi} \right] + \frac{\partial}{\partial \xi} \left[ B_{zz} \frac{\partial \bar{\varphi}_y}{\partial \xi} \right] + \frac{S(\xi)}{\mu} \left( \frac{\partial \bar{v}}{\partial \xi} - \bar{\varphi}_y \right) \\ & - \eta R_{yz}(\xi) \frac{\partial^2 \bar{\varphi}_z}{\partial \tau^2} - \eta R_{zz}(\xi) \frac{\partial^2 \bar{\varphi}_y}{\partial \tau^2} = \tilde{q}(\xi, \tau), \quad \xi \in (0, 1), \end{aligned} \quad (34)$$

where

$$\begin{aligned} \bar{p}(\xi, \tau) &= P(\xi, \tau) - m(\xi) \sum_{i=1}^8 \frac{d^2 \bar{F}_i}{d\tau^2} g_i(\xi), \\ \bar{q}(\xi, \tau) &= Q(\xi, \tau) - m(\xi) \sum_{i=1}^8 \frac{d^2 \bar{F}_i}{d\tau^2} \bar{g}_i(\xi), \\ \tilde{p}(\xi, \tau) &= \eta \sum_{i=1}^8 \frac{d^2 \bar{F}_i}{d\tau^2} [R_{yy}(\xi) h_i(\xi) + R_{yz}(\xi) \bar{h}_i(\xi)], \\ \tilde{q}(\xi, \tau) &= \eta \sum_{i=1}^8 \frac{d^2 \bar{F}_i}{d\tau^2} [R_{yz}(\xi) h_i(\xi) + R_{zz}(\xi) \bar{h}_i(\xi)] \end{aligned} \quad (35)$$

and the associated homogeneous boundary conditions:

At  $\xi = 0$ :

$$-\gamma_{12} \left( B_{yy} \frac{\partial \bar{\varphi}_z}{\partial \xi} + B_{yz} \frac{\partial \bar{\varphi}_y}{\partial \xi} \right) + \gamma_{11} \bar{\varphi}_z = 0, \quad (36)$$

$$-\gamma_{22} \frac{S}{\mu} \left( \frac{\partial \bar{w}}{\partial \xi} - \bar{n}_z \right) + \gamma_{21} \bar{w} = 0, \quad (37)$$



$$-\gamma_{32} \left( B_{yz} \frac{\partial \bar{\varphi}_z}{\partial \xi} + B_{zz} \frac{\partial \bar{\varphi}_y}{\partial \xi} \right) + \gamma_{31} \bar{\varphi}_y = 0, \quad (38)$$

$$-\gamma_{42} \frac{S}{\mu} \left( \frac{\partial \bar{v}}{\partial \xi} - \bar{\varphi}_y \right) + \gamma_{41} \bar{v} = 0. \quad (39)$$

At  $\xi = 1$ :

$$\gamma_{52} \left( B_{yy} \frac{\partial \bar{\varphi}_z}{\partial \xi} + B_{yy} \frac{\partial \bar{\varphi}_y}{\partial \xi} \right) + \gamma_{51} \bar{\varphi}_z = 0, \quad (40)$$

$$\gamma_{62} \frac{S}{\mu} \left( \frac{\partial \bar{w}}{\partial \xi} - \bar{\varphi}_z \right) + \gamma_{61} \bar{w} = 0, \quad (41)$$

$$\gamma_{72} \left( B_{yz} \frac{\partial \bar{\varphi}_z}{\partial \xi} + B_{zz} \frac{\partial \bar{\varphi}_y}{\partial \xi} \right) + \gamma_{71} \bar{\varphi}_z = 0, \quad (42)$$

$$\gamma_{82} \frac{S}{\mu} \left( \frac{\partial \bar{v}}{\partial \xi} - \bar{\varphi}_y \right) + \gamma_{81} \bar{v} = 0. \quad (43)$$

The transformed initial conditions (17) become

$$\begin{aligned} \bar{w}(\xi, 0) &= W_0(\xi) - \sum_{i=1}^8 \bar{F}_i(0) g_i(\xi), & \bar{v}(\xi, 0) &= V_0(\xi) - \sum_{i=1}^8 \bar{F}_i(0) \bar{g}_i(\xi), \\ \frac{\partial \bar{w}(\xi, 0)}{\partial \tau} &= \dot{W}_0(\xi) - \sum_{i=1}^8 \frac{d\bar{F}_i(0)}{d\tau} g_i(\xi), & \frac{\partial \bar{v}(\xi, 0)}{\partial \tau} &= \dot{V}_0(\xi) - \sum_{i=1}^8 \frac{d\bar{F}_i(0)}{d\tau} \bar{g}_i(\xi), \\ \bar{\varphi}_z(\xi, 0) &= \Psi_{z0}(\xi) - \sum_{i=1}^8 \bar{F}_i(0) h_i(\xi), & \bar{\varphi}_y(\xi, 0) &= \Psi_{y0}(\xi) - \sum_{i=1}^8 \bar{F}_i(0) \bar{h}_i(\xi), \\ \frac{\partial \bar{\varphi}_z(\xi, 0)}{\partial \tau} &= \dot{\Psi}_{z0}(\xi) - \sum_{i=1}^8 \frac{d\bar{F}_i(0)}{d\tau} h_i(\xi), & \frac{\partial \bar{\varphi}_y(\xi, 0)}{\partial \tau} &= \dot{\Psi}_{y0}(\xi) - \sum_{i=1}^8 \frac{d\bar{F}_i(0)}{d\tau} \bar{h}_i(\xi). \end{aligned} \quad (44)$$

### 3.2. SHIFTING FUNCTIONS AND ITS PHYSICAL MEANINGS

The system composed of equations (19–30) in terms of the shifting functions presents the static problem of a pretwisted non-uniform Timoshenko beam subjected to unit end restraints. The shifting functions  $g_i$ ,  $\bar{g}_i$ ,  $h_i$  and  $\bar{h}_i$  are the static deflections and angle of rotation due to bending in the  $z$  and  $y$  direction, respectively, of a generally elastically restrained pretwisted beam subjected to a unit transformed moment or a unit transformed shear force at the ends respectively. When the rotational spring constant is infinity or zero, the unit transformed moment is a unit end slope or a unit end moment. When the translational spring constant is infinity or zero, the unit transformed shear force is a unit end displacement or a unit end shear force.

The corresponding shear forces are obtained by integrating the governing equations (19) and (21) once, respectively,

$$\bar{Q}_{z,i}(\zeta) = \frac{S(\zeta)}{\mu} \left( \frac{dg_i}{d\zeta} - h_i \right) = c_{1,i}, \quad (45)$$

$$\bar{Q}_{y,i}(\zeta) = \frac{S(\zeta)}{\mu} \left( \frac{d\bar{g}_i}{d\zeta} - \bar{h}_i \right) = c_{2,i}. \quad (46)$$

Substituting equations (45) and (46) into equations (20) and (22), respectively, and integrating these once, one obtains

$$-\bar{M}_{z,i}(\zeta) = B_{yy} \frac{dh_i}{d\zeta} + B_{yz} \frac{d\bar{h}_i}{d\zeta} = -c_{1,i}\zeta + c_{3,i}, \quad (47)$$

$$-\bar{M}_{y,i}(\zeta) = B_{yz} \frac{dh_i}{d\zeta} + B_{zz} \frac{d\bar{h}_i}{d\zeta} = -c_{2,i}\zeta + c_{4,i}. \quad (48)$$

Obviously, the coefficients  $c_{1,i}$ ,  $c_{2,i}$ ,  $c_{3,i}$  and  $c_{4,i}$  are the corresponding shear forces and moments at  $\zeta = 0$  in the direction of  $z$  and  $y$  respectively. The following equations can be obtained easily via equations (47) and (48):

$$(B_{zz}B_{yy} - B_{yz}^2) \frac{dh_i}{d\zeta} = -c_{1,i}\zeta B_{zz} + c_{2,i}\zeta B_{yz} + c_{3,i}B_{zz} - c_{4,i}B_{yz}, \quad (49)$$

$$(B_{yz}^2 - B_{zz}B_{yy}) \frac{d\bar{h}_i}{d\zeta} = -c_{1,i}\zeta B_{yz} + c_{2,i}\zeta B_{yy} + c_{3,i}B_{yz} - c_{4,i}B_{yy}. \quad (50)$$

Integrating equations (49) and (50) once, respectively, one obtains

$$h_i = c_{1,i}\phi_1(\zeta) + c_{2,i}\phi_2(\zeta) + c_{3,i}\phi_3(\zeta) + c_{4,i}\phi_4(\zeta) + c_{5,i}, \quad (51)$$

$$\bar{h}_i = c_{1,i}\varphi_1(\zeta) + c_{2,i}\varphi_2(\zeta) + c_{3,i}\varphi_3(\zeta) + c_{4,i}\varphi_4(\zeta) + c_{6,i}, \quad (52)$$

where

$$\phi_1(\zeta) = \int_0^\zeta \frac{-\zeta B_{zz}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)} d\zeta,$$

$$\phi_2(\zeta) = \int_0^\zeta \frac{-\zeta B_{yz}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)} d\zeta,$$

$$\phi_3(\zeta) = \int_0^\zeta \frac{B_{zz}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)} d\zeta,$$

$$\phi_4(\zeta) = \int_0^\zeta \frac{-B_{yz}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)} d\zeta,$$

$$\begin{aligned}\varphi_1(\xi) &= \phi_2(\xi), \quad \varphi_3(\xi) = \phi_4(\xi), \\ \varphi_2(\xi) &= \int_0^\xi \frac{-\zeta B_{yy}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)} d\zeta, \\ \varphi_4(\xi) &= \int_0^\xi \frac{B_{yy}(\zeta)}{B_{zz}(\zeta)B_{yy}(\zeta) - B_{yz}^2(\zeta)} d\zeta.\end{aligned}\quad (53)$$

Substituting equations (51), (52) and (53) back into equations (45) and (46), respectively, the following displacements are obtained:

$$g_i(\xi) = c_{1,i}w_1(\xi) + c_{2,i}w_2(\xi) + c_{3,i}w_3(\xi) + c_{4,i}w_4(\xi) + c_{5,i}\xi + c_{7,i}, \quad (54)$$

$$\bar{g}_i = c_{1,i}v_1(\xi) + c_{2,i}v_2(\xi) + c_{3,i}v_3(\xi) + c_{4,i}v_4(\xi) + c_{6,i}\xi + c_{8,i}, \quad (55)$$

where  $c_{5,i}$ ,  $c_{6,i}$ ,  $c_{7,i}$  and  $c_{8,i}$  are  $h_i(0)$ ,  $\bar{h}_i(0)$ ,  $g_i(0)$  and  $\bar{g}_i(0)$ , respectively,

$$\begin{aligned}w_1(\xi) &= \int_0^\xi \left( \frac{\mu}{S(\zeta)} + \phi_1(\zeta) \right) d\zeta, & w_i(\xi) &= \int_0^\xi \phi_i(\zeta) d\zeta, \quad i = 2, 3, 4, \\ v_2(\xi) &= \int_0^\xi \left( \frac{\mu}{S(\zeta)} + \varphi_2(\zeta) \right) d\zeta, & v_i(\xi) &= \int_0^\xi \varphi_i(\zeta) d\zeta, \quad i = 1, 3, 4.\end{aligned}\quad (56)$$

Substituting equations (45–48, 51–56) into the boundary conditions (23–30), the coefficients  $c_{i,j}$ ,  $i, j = 1, 2, \dots, 8$ , of the general shifting solutions (51–52, 54–55) are obtained:

$$\begin{bmatrix} c_{1,i} \\ c_{2,i} \\ c_{3,i} \\ c_{4,i} \\ c_{5,i} \\ c_{6,i} \\ c_{7,i} \\ c_{8,i} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\gamma_{12} & 0 & \gamma_{11} & 0 & 0 & 0 \\ -\gamma_{22} & 0 & 0 & 0 & 0 & 0 & \gamma_{21} & 0 \\ 0 & 0 & 0 & \gamma_{32} & 0 & \gamma_{31} & 0 & 0 \\ 0 & -\gamma_{42} & 0 & 0 & 0 & 0 & 0 & \gamma_{41} \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \gamma_{51} & 0 & 0 & 0 \\ \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \gamma_{61} & 0 & \gamma_{61} & 0 \\ \alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12} & 0 & \gamma_{71} & 0 & 0 \\ \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & 0 & \gamma_{81} & 0 & \gamma_{81} \end{bmatrix}^{-1} \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ \delta_{i4} \\ \delta_{i5} \\ \delta_{i6} \\ \delta_{i7} \\ \delta_{i8} \end{bmatrix}, \quad (57)$$

where

$$\begin{aligned}\alpha_1 &= \gamma_{51}\phi_1(1) - \gamma_{52}, & \alpha_2 &= \gamma_{51}\phi_2(1), & \alpha_3 &= \gamma_{51}\phi_3(1) + \gamma_{52}, \\ \alpha_4 &= \gamma_{51}\phi_4(1), & \alpha_5 &= \gamma_{61}w_1(1) + \gamma_{62}, & \alpha_6 &= \gamma_{61}w_2(1), \\ \alpha_7 &= \gamma_{61}w_3(1), & \alpha_8 &= \gamma_{61}w_4(1), & \alpha_9 &= \gamma_{71}\varphi_1(1), \\ \alpha_{10} &= \gamma_{71}\varphi_2(1) - \gamma_{72}, & \alpha_{11} &= \gamma_{71}\varphi_3(1), & \alpha_{12} &= \gamma_{71}\varphi_4(1) + \gamma_{72}, \\ \alpha_{13} &= \gamma_{81}v_1(1), & \alpha_{14} &= \gamma_{81}v_2(1) + \gamma_{82}, & \alpha_{15} &= \gamma_{81}v_3(1), \\ \alpha_{16} &= \gamma_{81}v_4(1).\end{aligned}\quad (58)$$

It should be noted that if the ratio of bending rigidity to shear rigidity  $\mu$  is zero,  $dg_i/d\xi = h_i$ ,  $d\bar{g}_i/d\xi = \bar{h}_i$  and the shifting functions become the same as those for Bernoulli–Euler beams given by Lin [18].

### 3.3. ORTHOGONALITY CONDITION

The solution for equations (26–37),  $\bar{w}(\xi, \tau)$ ,  $\bar{v}(\xi, \tau)$ ,  $\bar{\varphi}_y(\xi, \tau)$  and  $\bar{\varphi}_z(\xi, \tau)$  can be obtained by using the method of eigenfunction expansion. The eigenfunctions are specified by the associated homogeneous governing differential equations and homogeneous boundary conditions.

To derive the orthogonality condition of the eigenfunctions of the system, one lets  $A_n^2$  be the  $n$ th eigenvalue or the square of the  $n$ th dimensionless natural frequency and  $[w_n \ \varphi_{zn} \ v_n \ \varphi_{yn}]^T$  be the  $n$ th eigenfunction of the system, where the superscript T is the symbol of transpose of a matrix. The governing characteristic differential equation can be expressed as

$$\{[\tilde{I}] + A_n^2[\tilde{M}]\} \begin{bmatrix} w_n \\ n_{zn} \\ v_n \\ n_{yn} \end{bmatrix} = 0, \quad (59)$$

where the differential operators  $[\tilde{I}]$  and  $[\tilde{M}]$  are

$$[\tilde{I}] = \begin{bmatrix} \frac{d}{d\xi} \left( \frac{S}{\mu} \frac{d}{d\xi} \right) & -\frac{d}{d\xi} \left( \frac{S}{\mu} \right) & 0 & 0 \\ \frac{S}{\mu} \frac{d}{d\xi} & \frac{d}{d\xi} \left( B_{yy} \frac{d}{d\xi} \right) - \frac{S}{\mu} & 0 & \frac{d}{d\xi} \left( B_{yz} \frac{d}{d\xi} \right) \\ 0 & 0 & \frac{d}{d\xi} \left( \frac{S}{\mu} \frac{d}{d\xi} \right) & -\frac{d}{d\xi} \left( \frac{S}{\mu} \right) \\ 0 & \frac{d}{d\xi} \left( B_{yz} \frac{d}{d\xi} \right) & \frac{S}{\mu} & \frac{d}{d\xi} \left( B_{zz} \frac{d}{d\xi} \right) - \frac{S}{\mu} \end{bmatrix} \quad (60)$$

and

$$[\tilde{M}] = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & \eta R_{yy} & 0 & \eta R_{yz} \\ 0 & 0 & m & 0 \\ 0 & \eta R_{yz} & 0 & \eta R_{zz} \end{bmatrix}, \quad A_n^2 = \frac{\rho(0)A(0)\Omega_n^2 L^4}{E(0)I_{yy}(0)}, \quad (61)$$

in which  $\Omega_n$  is the  $n$ th natural frequency. The eigenfunctions satisfy the boundary conditions (36–43). It can be observed that equation (59) and the associated boundary conditions (36–43) take the meaning of the free vibration of an elastically restrained non-uniform beam. The eigenfunctions and the eigenvalues can be obtained by using the modified transfer

matrix method proposed by Lin *et al.* [26]. It was shown that as the element of the field transfer matrix can be integrated anaytically, the exact field transfer matrix of the system is, therefore, obtained.

Taking the inner product, one can easily show that

$$\int_0^1 [w_j \ \varphi_{zj} \ v_j \ \varphi_{yj}] [\tilde{M}] \begin{bmatrix} w_n \\ \varphi_{zn} \\ v_n \\ \varphi_{yn} \end{bmatrix} d\zeta = \int_0^1 [w_n \ \varphi_{zn} \ v_n \ \varphi_{yn}] [\tilde{M}] \begin{bmatrix} w_j \\ \varphi_{zj} \\ v_j \\ \varphi_{yj} \end{bmatrix} d\zeta \quad (62)$$

and

$$\int_0^1 [w_j \ \varphi_{zj} \ v_j \ \varphi_{yj}] [\tilde{\Gamma}] \begin{bmatrix} w_n \\ \varphi_{zn} \\ v_n \\ \varphi_{yn} \end{bmatrix} d\zeta = \int_0^1 [w_n \ \varphi_{zn} \ v_n \ \varphi_{yn}] [\tilde{\Gamma}] \begin{bmatrix} w_j \\ \varphi_{zj} \\ v_j \\ \varphi_{yj} \end{bmatrix} d\zeta + \tilde{B}, \quad (63)$$

where

$$\begin{aligned} \tilde{B} = & w_j \frac{S}{\mu} \left( \frac{dw_n}{d\zeta} - \varphi_{zn} \right) \Big|_0^1 - w_n \frac{S}{\mu} \left( \frac{dw_j}{d\zeta} - \varphi_{zj} \right) \Big|_0^1 \\ & + \varphi_{zj} \left[ B_{yy} \frac{d\varphi_{zn}}{d\zeta} + B_{yz} \frac{d\varphi_{yn}}{d\zeta} \right] \Big|_0^1 - \varphi_{zn} \left[ B_{yy} \frac{d\varphi_{zj}}{d\zeta} + B_{yz} \frac{d\varphi_{yj}}{d\zeta} \right] \Big|_0^1 \\ & + \varphi_{yj} \left[ B_{yz} \frac{d\varphi_{zn}}{d\zeta} + B_{zz} \frac{d\varphi_{yn}}{d\zeta} \right] \Big|_0^1 - \varphi_{yn} \left[ B_{yz} \frac{d\varphi_{zj}}{d\zeta} + B_{zz} \frac{d\varphi_{yj}}{d\zeta} \right] \Big|_0^1 \\ & + v_j \frac{S}{\mu} \left( \frac{dv_n}{d\zeta} - \varphi_{yn} \right) \Big|_0^1 - v_n \frac{S}{\mu} \left( \frac{dv_j}{d\zeta} - \varphi_{yj} \right) \Big|_0^1 \end{aligned} \quad (64)$$

and  $\tilde{B}$  vanishes because of the boundary conditions (29–36). Thus, the self-adjointness of the system is proved. Consequently, the orthogonality condition is obtained as follows:

$$\begin{aligned} & \int_0^1 \{ m(w_j w_n + v_j v_n) + \eta R_{yy} \varphi_{zn} \varphi_{zj} + \eta R_{zz} \varphi_{yn} \varphi_{yj} + \eta R_{yz} (\varphi_{zn} \varphi_{yj} + \varphi_{yn} \varphi_{zj}) \} d\zeta \\ & = \begin{cases} 0, & j \neq n, \\ \varepsilon_n, & j = n, \end{cases} \end{aligned} \quad (65)$$

where  $\varepsilon_n$  is a real number.

## 3.4. MODE SUPERPOSITION

The solution  $\bar{w}(\xi, \tau)$ ,  $\bar{v}(\xi, \tau)$ ,  $\bar{\varphi}_z(\xi, \tau)$  and  $\bar{\varphi}_y(\xi, \tau)$  specified by equations (31–44) can be expressed in the following eigenfunction expansion form:

$$\begin{bmatrix} \bar{w}(\xi, \tau) \\ \bar{\varphi}_z(\xi, \tau) \\ \bar{v}(\xi, \tau) \\ \bar{\varphi}_y(\xi, \tau) \end{bmatrix} = \sum_{n=1}^{\infty} T_n(\tau) \begin{bmatrix} w_n(\xi) \\ \varphi_{zn}(\xi) \\ v_n(\xi) \\ \varphi_{yn}(\xi) \end{bmatrix}. \quad (66)$$

Substituting it back into the governing equations (31–34) and the initial conditions (44), multiplying by  $[w_n(\xi) \varphi_{zn}(\xi) v_n(\xi) \varphi_{yn}(\xi)]$  and integrating in accordance with the orthogonality condition (65), one obtains

$$\frac{d^2 T_n}{d\tau^2} + A_n^2 T_n = p_n^*, \quad (67)$$

where  $p_n^*(\xi) = (1/\varepsilon_n) \int_0^1 [w_n \bar{p} - \varphi_{zn} \bar{p} + v_n \bar{q} - \varphi_{yn} \bar{q}] d\xi$ . The corresponding initial conditions are

$$\begin{aligned} T_n(0) = \int_0^1 \{ & m(w_n \bar{w}(\xi, 0) + v_n \bar{v}(\xi, 0)) + \eta R_{yy} \varphi_{zn} \bar{\varphi}_z(\xi, 0) + \eta R_{zz} \varphi_{yn} \bar{\varphi}_y(\xi, 0) \\ & + \eta R_{yz} [\varphi_{zn} \bar{\varphi}_y(\xi, 0) + \varphi_{yn} \bar{\varphi}_z(\xi, 0)] \} d\xi, \end{aligned} \quad (68)$$

$$\begin{aligned} \frac{dT_n(0)}{d\tau} = \int_0^1 \left\{ & m \left( w_n \frac{\partial \bar{w}(\xi, 0)}{\partial \tau} + v_n \frac{\partial \bar{v}(\xi, 0)}{\partial \tau} \right) + \eta R_{yy} \varphi_{zn} \frac{\partial \bar{\varphi}_z(\xi, 0)}{\partial \tau} \right. \\ & \left. + \eta R_{zz} \varphi_{yn} \frac{\partial \bar{\varphi}_y(\xi, 0)}{\partial \tau} + \eta R_{yz} \left[ \varphi_{zn} \frac{\partial \bar{\varphi}_y(\xi, 0)}{\partial \tau} + \varphi_{yn} \frac{\partial \bar{\varphi}_z(\xi, 0)}{\partial \tau} \right] \right\} d\xi. \end{aligned} \quad (69)$$

The solution of equation (67) is

$$T_n(\tau) = T_n(0) \cos A_n \tau + \frac{1}{A_n} \frac{dT_n(0)}{d\tau} \sin A_n \tau + \frac{1}{A_n} \int_0^\tau p_n^*(\xi) \sin A_n(\tau - \xi) d\xi. \quad (70)$$

After substituting solution (70) back into equation (66), the general forced response of the beam with time-dependent boundary conditions is finally obtained by substituting the shifting functions (51–52, 54–55) and equation (66) into equation (18).

## 4. BOUNDARY CONTROL

Consider the steady response of a pretwisted Timoshenko beam subjected to the harmonic concentrated transverse loads and the harmonic boundary excitation forces. The

external concentrated forces and boundary inputs are

$$\begin{aligned} P(\xi, \tau) &= P^* \delta(\xi - \xi_0) \sin \varpi \tau, \quad Q(\xi, \tau) = Q^* \delta(\xi - \xi_0) \sin \varpi \tau, \\ \bar{F}_i(\tau) &= \tilde{F}_i \sin \varpi \tau, \quad i = 1, 2, \dots, 8. \end{aligned} \quad (71)$$

If the transient response from the initial conditions is neglected, the general dynamic solution (18) is reduced into the following steady solution:

$$\begin{aligned} V(\xi, \tau) &= \tilde{V}(\xi) \sin \varpi \tau = \left( P^* V_p^*(\xi) + Q^* V_q^*(\xi) + \sum_{i=1}^8 \tilde{F}_i V_i^*(\xi) \right) \sin \varpi \tau, \\ W(\xi, \tau) &= \tilde{W}(\xi) \sin \varpi \tau = \left( P^* W_p^*(\xi) + Q^* W_q^*(\xi) + \sum_{i=1}^8 \tilde{F}_i W_i^*(\xi) \right) \sin \varpi \tau, \\ \Psi_y(\xi, \tau) &= \tilde{\Psi}_y(\xi) \sin \varpi \tau = \left( P^* \Psi_{yp}^*(\xi) + Q^* \Psi_{yq}^*(\xi) + \sum_{i=1}^8 \tilde{F}_i \Psi_{yi}^*(\xi) \right) \sin \varpi \tau, \\ \Psi_z(\xi, \tau) &= \tilde{\Psi}_z(\xi) \sin \varpi \tau = \left( P^* \Psi_{zp}^*(\xi) + Q^* \Psi_{zq}^*(\xi) + \sum_{i=1}^8 \tilde{F}_i \Psi_{zi}^*(\xi) \right) \sin \varpi \tau, \end{aligned} \quad (72)$$

where

$$\begin{aligned} V_p^*(\xi) &= \sum_{n=1}^{\infty} \frac{w_n(\xi_0) v_n(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \quad V_q^*(\xi) = \sum_{n=1}^{\infty} \frac{v_n(\xi_0) v_n(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \\ V_i^*(\xi) &= \bar{g}_i(\xi) + \sum_{n=1}^{\infty} \frac{\varpi^2 c_{i,n} v_n(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \\ W_p^*(\xi) &= \sum_{n=1}^{\infty} \frac{w_n(\xi_0) w_n(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \quad W_q^*(\xi) = \sum_{n=1}^{\infty} \frac{v_n(\xi_0) w_n(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \\ W_i^*(\xi) &= \bar{g}_i(\xi) + \sum_{n=1}^{\infty} \frac{\varpi^2 c_{i,n} w_n(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \\ \Psi_{yp}^*(\xi) &= \sum_{n=1}^{\infty} \frac{w_n(\xi_0) \Psi_{yn}(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \quad \Psi_{yq}^*(\xi) = \sum_{n=1}^{\infty} \frac{v_n(\xi_0) \Psi_{yn}(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \\ \Psi_{yi}^*(\xi) &= \bar{h}_i(\xi) + \sum_{n=1}^{\infty} \frac{\varpi^2 c_{i,n} \Psi_{yn}(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \\ \Psi_{zp}^*(\xi) &= \sum_{n=1}^{\infty} \frac{w_n(\xi_0) \Psi_{zn}(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \quad \Psi_{zq}^*(\xi) = \sum_{n=1}^{\infty} \frac{v_n(\xi_0) \Psi_{zn}(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \\ \Psi_{zi}^*(\xi) &= \bar{h}_i(\xi) + \sum_{n=1}^{\infty} \frac{\varpi^2 c_{i,n} \Psi_{zn}(\xi)}{\varepsilon_n (\Lambda_n^2 - \varpi^2)}, \end{aligned} \quad (73)$$

in which

$$c_{i,n} = \int_0^1 m[w_n g_i + v_n \bar{g}_i] d\xi + \eta \int_0^1 \{\varphi_{zn}[R_{yy}h_i + R_{yz}\bar{h}_i] + \varphi_{zn}[R_{yy}h_i + R_{yz}\bar{h}_i]\} d\xi, \\ i = 1, 2, \dots, 8. \quad (74)$$

If the displacements  $V$  and  $W$  at  $\xi = \xi_1$  are controlled to be zero, the following two conditions are obtained from equation (72):

$$\sum_{i=1}^8 \tilde{F}_i W_i^*(\xi_1) = -P^* W_p^*(\xi_1) - Q^* W_q^*(\xi_1), \\ \sum_{i=1}^8 \tilde{F}_i V_i^*(\xi_1) = -P^* V_p^*(\xi_1) - Q^* V_q^*(\xi_1). \quad (75)$$

One can choose only two boundary inputs to satisfy the conditions. If the  $m$ th and  $n$ th inputs are chosen, the corresponding coefficients of the boundary inputs are obtained as

$$\begin{bmatrix} \tilde{F}_m \\ \tilde{F}_n \end{bmatrix} = - \begin{bmatrix} W_m^*(\xi_1) & W_n^*(\xi_1) \\ V_m^*(\xi_1) & V_n^*(\xi_1) \end{bmatrix}^{-1} \begin{bmatrix} P^* W_p^*(\xi_1) + Q^* W_q^*(\xi_1) \\ P^* V_p^*(\xi_1) + Q^* V_q^*(\xi_1) \end{bmatrix}, \quad \tilde{F}_i = 0, i \neq m \text{ and } n. \quad (76)$$

Based on these results, the energy required for the displacement control can be derived easily.

## 5. NUMERICAL RESULTS AND DISCUSSION

To illustrate the application of the method and explore the physical phenomena of the system, the following examples are presented.

**Example 1.** To establish the element stiffness matrix of a non-uniform beam with arbitrary pretwist, the static deflection curves of the beam subjected only to a unit displacement or a unit slope at either end of the beam segment have to be determined. However, it is known from the meanings of the shifting functions explored in the previous section that these deflection curves are just the shifting functions  $g_i(\xi)$ ,  $\bar{g}_i(\xi)$ ,  $h_i(\xi)$ ,  $\bar{h}_i(\xi)$  for the clamped-clamped beam listed in the case 2 of Appendix A. Thus, the element stiffness matrix relation can be written as

$$\begin{bmatrix} -Q_z(0) & M_z(0) & Q_z(1) & -M_z(1) & -Q_y(0) & M_y(0) & Q_y(1) & -M_y(1) \end{bmatrix}^T \\ = [k_{ij}]_{8 \times 8} [\phi_z^*(0) \ w^*(0) \ \phi_y^*(0) \ v^*(0) \ \phi_z^*(1) \ w^*(1) \ \phi_y^*(1) \ v^*(1)]^T, \quad (77)$$

where  $v^*$ ,  $w^*$ ,  $\phi_y^*$ , and  $\phi_z^*$  represent the static displacements and the angle of rotation due to bending in the  $y$  and  $z$  directions, respectively, and the elements of the stiffness matrix are

$$k_{1i} = -c_{1,i}, k_{2i} = c_{3,i}, k_{3i} = c_{1,i}, k_{4i} = -c_{1,i} + c_{3,i}, \\ k_{5i} = -c_{2,i}, k_{6i} = -c_{4,i}, k_{7i} = c_{2,i}, k_{8i} = -c_{2,i} + c_{4,i}, \quad (78)$$



TABLE 1

The influence of the tip pretwist angle  $\Phi$ , the rotatory inertia and shear deformation on the transverse displacements of cantilever beams subjected to a harmonic concentrated transverse force and boundary inputs

$\xi$	$\Phi = 30^\circ$				$\Phi = 90^\circ$			
	$\eta = 0,$ $\mu = 0$		$\eta = 0.001,$ $\mu = 0.003133$		$\eta = 0,$ $\mu = 0$		$\eta = 0.001,$ $\mu = 0.003133$	
	$W$	$V$	$W$	$V$	$W$	$V$	$W$	$V$
0.0	0.010	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100
0.1	0.0118	0.0100	0.0119	0.0100	0.0123	0.0100	0.0124	0.0100
0.2	0.0172	0.0100	0.0175	0.0100	0.0191	0.0102	0.0194	0.0102
0.3	0.0262	0.0101	0.0269	0.0101	0.0302	0.0107	0.0309	0.0107
0.4	0.0389	0.0101	0.0400	0.0102	0.0451	0.0116	0.0463	0.0117
0.5	0.0552	0.0103	0.0569	0.0103	0.0633	0.0131	0.0651	0.0132
0.6	0.0753	0.0105	0.0778	0.0105	0.0843	0.0151	0.0866	0.0153
0.7	0.0993	0.0108	0.1026	0.0108	0.1074	0.0178	0.1104	0.0181
0.8	0.1276	0.0122	0.1318	0.0113	0.1322	0.0211	0.1358	0.0215
0.9	0.1604	0.0118	0.1658	0.0119	0.1581	0.0249	0.1623	0.0255
1.0	0.1987	0.0126	0.2054	0.0127	0.1849	0.0292	0.1896	0.0301

Note: Transverse force and boundary inputs are given as follows:  $[P(\xi, \tau) = 0.01\delta(\xi - 0.5) \sin \tau, Q(\xi, \tau) = 0, \bar{F}_2 = \bar{F}_4 = 0.01 \sin \tau, \bar{F}_i(\tau) = 0, i = 1, 3, 5, \dots, 8, B_{yy} = (1 - 0.1\xi) \cos^2 \xi\Phi + 20(1 - 0.1\xi)^3 \sin^2 \xi\Phi, B_{zz} = 20(1 - 0.1\xi)^3 \cos^2 \xi\Phi + (1 - 0.1\xi) \sin^2 \xi\Phi, B_{yz} = [10(1 - 0.1\xi)^3 - 0.5(1 - 0.1\xi)] \sin 2\xi\Phi]$ .

in which the coefficients  $c_{j,i}$  are the coefficients of the shifting functions. If the functions in equations (53) and (56) can be integrated analytically, the exact stiffness matrix is obtained. Otherwise, an accurate solution can be easily obtained by using the numerical integration method. If the shear deformation is neglected, the matrix relation (77) becomes the same as that given by Lin [18].

For the following numerical solutions, the first five modes are used in the series expansion. Table 1 shows the influence of the tip pretwist angle  $\Phi$ , the rotatory inertia constant  $\eta$ , and the shear deformation constant  $\mu$  on the transverse displacements of a cantilever beam subjected to harmonic concentrated transverse load  $P$  and the boundary excitations  $\bar{F}_2$  and  $\bar{F}_4$ . It is shown that larger the constants  $\eta$  and  $\mu$ , the greater is its influence on the displacements. When the tip pretwist angle  $\Phi$  is increased, the displacement  $W$  is increased but the displacement  $V$  is decreased. This is the reason why only the transverse load  $P$  in  $z$  direction is applied to the beam.

**Example 2.** Consider a sliding-free beam subjected to only the transverse load  $P$ . Letting  $m = 2$  and  $n = 4$  and taking the relation (76), the tip-displacements  $W(1, \tau)$  and  $V(1, \tau)$  can be controlled to be zero. The displacements  $W$  and  $V$  of the beam subjected to a concentrated load  $p$  at the center position are shown in Figure 2. The energy required to control the performance of the beam is

$$\tilde{E}(\tau) = E^* \sin^2 \omega \tau = (\tilde{F}_2 \tilde{Q}_z + \tilde{F}_4 \tilde{Q}_y) \sin^2 \omega \tau, \quad (79)$$

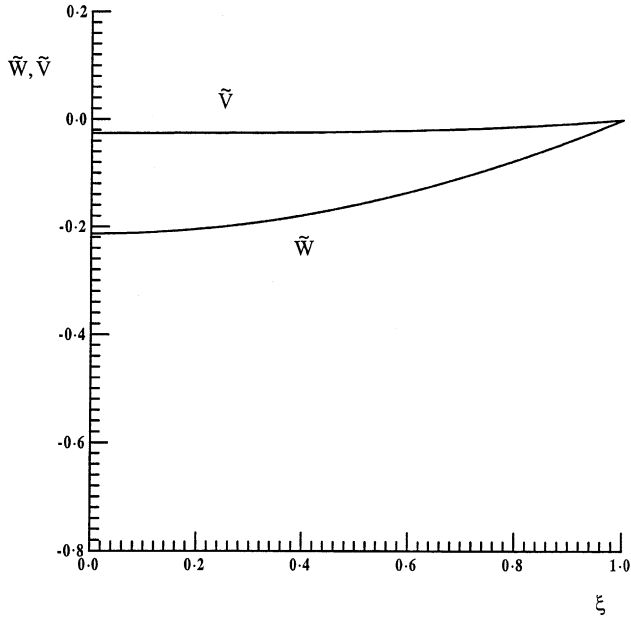


Figure 2. The transverse displacements of cantilever beams subjected to a harmonic concentrated transverse force and the boundary controls [ $P(\xi, \tau) = 0.01\delta(\xi-0.5)\sin\tau$ ,  $Q(\xi, \tau) = 0$ ,  $\bar{F}_2 = \bar{F}_2 \sin \tau$ ,  $\bar{F}_4 = \bar{F}_4 \sin \tau$ ,  $\bar{F}_i(\tau) = 0$ ,  $i = 1, 3, 5, \dots, 8$ ,  $B_{yy} = (1 - 0.1\xi)^4 \cos^2 \xi\pi/3 + 20(1 - 0.1\xi)^4 \sin^2 \xi\pi/3$ ,  $B_{zz} = 20(1 - 0.1\xi)^4 \cos^2 \xi\pi/3 + (1 - 0.1\xi)^4 \sin^2 \xi\pi/3$ ,  $B_{yz} = 9.5(1 - 0.1\xi)^4 \sin 2\xi\pi/3$ ,  $\eta = 0.001$ ,  $\mu = 0.003133$ ].

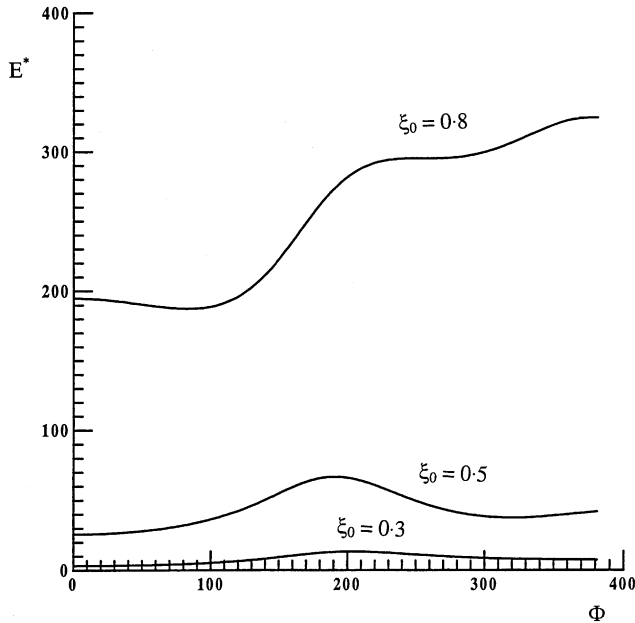


Figure 3. The influence of total pretwist angle and the position of the external loading on the control energy of a cantilever beam [ $P(\xi, \tau) = 0.01\delta(\xi-\xi_0)\sin\tau$ ,  $Q(\xi, \tau) = 0$ ,  $\bar{F}_2 = \bar{F}_2 \sin \tau$ ,  $\bar{F}_4 = \bar{F}_4 \sin \tau$ ,  $\bar{F}_i(\tau) = 0$ ,  $i = 1, 3, 5, \dots, 8$ ,  $B_{yy} = (1 - 0.1\xi)^4 \cos^2 \xi\Phi + 20(1 - 0.1\xi)^4 \sin^2 \xi\Phi$ ,  $B_{zz} = 20(1 - 0.1\xi)^4 \cos^2 \xi\Phi + (1 - 0.1\xi)^4 \sin^2 \xi\Phi$ ,  $B_{yz} = 9.5(1 - 0.1\xi)^4 \sin 2\xi\Phi$ ,  $\eta = 0.001$ ,  $\mu = 0.003133$ ].

where  $\{\tilde{F}_2, \tilde{F}_4\}$  and  $\{\tilde{Q}_y, \tilde{Q}_z\}$  are the input boundary displacements and the corresponding shear forces at  $\xi = 0$  respectively. The shear forces are

$$\tilde{Q}_y = \frac{S(0)}{\mu} \left( \frac{d\tilde{V}(0)}{d\xi} - \tilde{\Psi}_y(0) \right), \quad \tilde{Q}_z = \frac{S(0)}{\mu} \left( \frac{d\tilde{W}(0)}{d\xi} - \tilde{\Psi}_z(0) \right). \tag{80}$$

Figure 3 shows the influence of total pretwist angle and the position of the external loading on the control energy  $E^*$ . It is shown that when the position of the external loading approaches the free end, the required control energy is evidently larger. Moreover, the total pretwist angle is increased, the required control energy is increased.

**Example 3.** Consider a sliding-spring beam subjected to only the transverse load  $P$  at  $\xi = 0.6$ . Letting  $m = 2$  and  $n = 4$  and taking relation (76), the displacements  $W(0.5, \tau)$  and  $V(0.5, \tau)$  can be controlled to be zero. In Figure 4, the effects of the boundary spring constant  $\gamma_{61}$  and the total pretwist angle  $\Phi$  on the control energy are shown. When the spring constant  $\gamma_{61}$  is increased, the required control energy is decreased substantially. In Figure 5, the effects of the boundary spring constant  $\gamma_{81}$  and the total pretwist angle  $\Phi$  on the control energy are shown. If the spring constant  $\gamma_{81}$  is small, the influence of  $\gamma_{81}$  on the required control energy is small. When the spring constant  $\gamma_{81}$  approaches the value of one, the required control energy is decreased evidently. These results are obtained because the displacement  $w$  in the  $z$  direction is dominant.

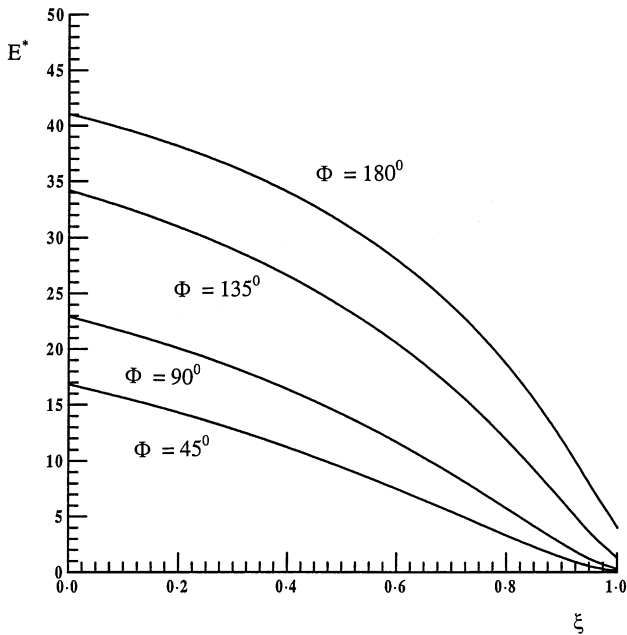


Figure 4. The influence of transverse spring constant  $\gamma_{61}$  and the total pretwist angle  $\Phi$  on the control energy of a beam  $[P(\xi, \tau) = 0.01\delta(\xi - 0.6) \sin \tau, Q(\xi, \tau) = 0, \tilde{F}_2 = \tilde{F}_2 \sin \tau, \tilde{F}_4 = \tilde{F}_4 \sin \tau, \tilde{F}_i(\tau) = 0, i = 1, 3, 5, \dots, 8, \gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = \gamma_{52} = \gamma_{72} = \gamma_{82} = 1, B_{yy} = (1 - 0.1\xi)^4 \cos^2 \xi\Phi + 20(1 - 0.1\xi)^4 \sin^2 \xi\Phi, B_{zz} = 20(1 - 0.1\xi)^4 \cos^2 \xi\Phi + (1 - 0.1\xi)^4 \sin^2 \xi\Phi, B_{yz} = 9.5(1 - 0.1\xi)^4 \sin 2\xi\Phi, \eta = 0.001, \mu = 0.003133]$ .

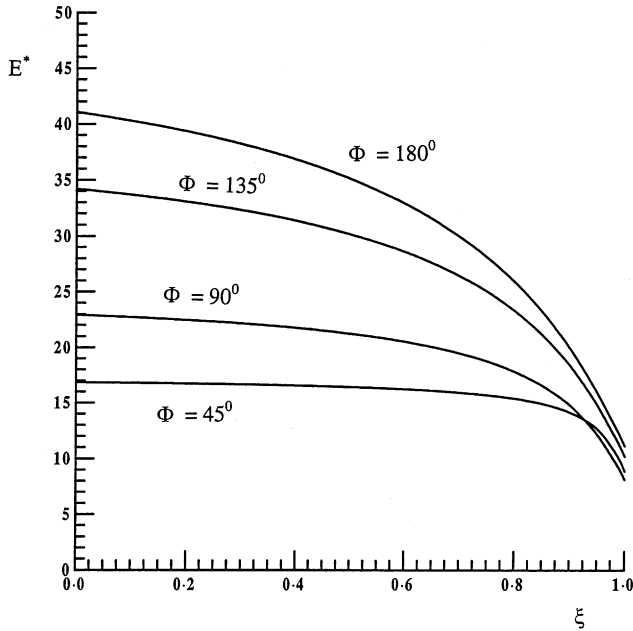


Figure 5. The influence of transverse spring constant  $\gamma_{81}$  and the total pretwist angle  $\Phi$  on the control energy of a beam [ $P(\xi, \tau) = 0.01\delta(\xi - 0.6) \sin \tau$ ,  $Q(\xi, \tau) = 0$ ,  $\bar{F}_2 = \bar{F}_2 \sin \tau$ ,  $\bar{F}_4 = \bar{F}_4 \sin \tau$ ,  $\bar{F}_i(\tau) = 0$ ,  $i = 1, 3, 5, \dots, 8$ ,  $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = \gamma_{52} = \gamma_{62} = \gamma_{72} = 1$ ,  $B_{yy} = (1 - 0.1\xi)^4 \cos^2 \xi\Phi + 20(1 - 0.1\xi)^4 \sin^2 \xi\Phi$ ,  $B_{zz} = 20(1 - 0.1\xi)^4 \cos^2 \xi\Phi + (1 - 0.1\xi)^4 \sin^2 \xi\Phi$ ,  $B_{yz} = 9.5(1 - 0.1\xi)^4 \sin 2\xi\Phi$ ,  $\eta = 0.001$ ,  $\mu = 0.003133$ ].

### 6. CONCLUSION

The governing differential equations with the general time-dependent elastic boundary conditions for the coupled bending–bending vibration of a pretwisted non-uniform beam are derived by Hamilton’s principle. An accurate solution procedure for the forced vibration of a pretwisted beam with general time-dependent elastic boundary conditions is proposed. The physical meanings of the shifting functions are revealed. The self-adjointness of the system is proved. The orthogonality condition for the eigenfunctions of a non-uniform pretwisted beam with elastic boundary conditions is derived. The stiffness matrix for a non-uniform beam with arbitrary pretwist is derived. The boundary control of a pretwist beam is derived. The effects of the total pretwist angle and the spring constants on the energy required to control the performance of a pretwisted beam are large.

### ACKNOWLEDGMENTS

The support of the National Science Council of Taiwan, R.O.C., is gratefully acknowledged (Grant number: Nsc88-2212-E168-003).

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#### APPENDIX: SHIFTING FUNCTIONS

*Case 1: Clamped–Free.* For this case,  $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = \gamma_{52} = \gamma_{62} = \gamma_{72} = \gamma_{82} = 1$ ,  $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} = \gamma_{51} = \gamma_{61} = \gamma_{71} = \gamma_{81} = 0$ , and the shifting functions are

$$g_1 = \xi, \quad g_2 = 1, \quad g_3 = g_4 = 0, \quad g_5 = w_3(\xi), \quad g_6 = w_1(\xi) + w_3(\xi),$$

$$\begin{aligned}
g_7 &= w_4(\xi), & g_8 &= w_2(\xi) + w_4(\xi), \\
\bar{g}_1 &= \bar{g}_2 = 0, \quad \bar{g}_3 = \xi, \quad \bar{g}_4 = 1, \quad \bar{g}_5 = v_3(\xi), \quad \bar{g}_6 = v_1(\xi) + v_3(\xi), \\
\bar{g}_7 &= v_4(\xi), & \bar{g}_8 &= v_2(\xi) + v_4(\xi); \\
h_1 &= 1, \quad h_2 = h_3 = h_4 = 0, & h_5 &= \phi_3(\xi), \quad h_6 = \phi_1(\xi) + \phi_3(\xi), \\
h_7 &= \phi_4(\xi), & h_8 &= \phi_2(\xi) + \phi_4(\xi), \\
\bar{h}_3 &= 1, \quad \bar{h}_1 = \bar{h}_2 = \bar{h}_4 = 0, & \bar{h}_5 &= \varphi_3(\xi), \quad \bar{h}_6 = \varphi_1(\xi) + \varphi_3(\xi) \\
\bar{h}_7 &= \varphi_4(\xi) & \bar{h}_8 &= \varphi_2(\xi) + \varphi_4(\xi).
\end{aligned}$$

*Case 2: Clamped-Clamped.* For this case,  $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = \gamma_{51} = \gamma_{61} = \gamma_{71} = \gamma_{81} = 1$ ,  $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} = \gamma_{52} = \gamma_{62} = \gamma_{72} = \gamma_{82} = 0$ , and the shifting functions are

$$\begin{aligned}
g_i(\xi) &= c_{1,i}w_1(\xi) + c_{2,i}w_2(\xi) + c_{3,i}w_3(\xi) + c_{4,i}w_4(\xi) + \delta_{i1}\xi + \delta_{i2}, \\
\bar{g}_i &= c_{1,i}v_1(\xi) + c_{2,i}v_2(\xi) + c_{3,i}v_3(\xi) + c_{4,i}v_4(\xi) + \delta_{i3}\xi + \delta_{i4}, \\
h_i &= c_{1,i}\phi_1(\xi) + c_{2,i}\phi_2(\xi) + c_{3,i}\phi_3(\xi) + c_{4,i}\phi_4(\xi) + \delta_{i1}, \\
\bar{h}_i &= c_{1,i}\varphi_1(\xi) + c_{2,i}\varphi_2(\xi) + c_{3,i}\varphi_3(\xi) + c_{4,i}\varphi_4(\xi) + \delta_{i3},
\end{aligned}$$

where

$$c_{ji} = d_{j1}(\delta_{i5} - \delta_{i1}) + d_{j2}(\delta_{i6} - \delta_{i1} - \delta_{i2}) + d_{j3}(\delta_{i7} - \delta_{i3}) + d_{j4}(\delta_{i8} - \delta_{i3} - \delta_{i4}),$$

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} = \begin{bmatrix} \phi_1(1) & \phi_2(1) & \phi_3(1) & \phi_4(1) \\ w_1(1) & w_2(1) & w_3(1) & w_4(1) \\ \varphi_1(1) & \varphi_2(1) & \varphi_3(1) & \varphi_4(1) \\ v_1(1) & v_2(1) & v_3(1) & v_4(1) \end{bmatrix}^{-1}.$$